R Function Tutorial in Module 8.2

Introduction to Computational Science:
Modeling and Simulation for the Sciences, 2nd Edition

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Introduction

In this chapter, we deal with models that are driven by the data. In such a situation, we have data measurements and wish to obtain a function that roughly goes through a plot of the data points capturing the trend of the data, or **fitting the data**. Subsequently, we can use the function to find estimates at places where data does not exist or to perform further computations. Moreover, determination of an appropriate fitting function can sometimes deepen our understanding of the reasons for the pattern of the data.

In this module, we consider several important functions, some which we have already used. By being familiar with basic functions and function transformations, the modeler can sometimes more readily fit a function to the data.

Linear Function

The concept of a linear function was essential in our discussions of the derivative and simulation techniques, such as Euler's Method. Here, we review some of the characteristics of functions whose graphs are lines.

The command in Quick Review Question 1 plots the graph of the linear function $y = 2t + 1$. This line has y-intercept 1, because $y = 1$ when $t = 0$. Thus, the graph crosses the y-axis when $t = 0$. With data measurements where t represents time, the y-intercept indicates the initial data value. The slope of this particular line is 2, which is the coefficient of t. Consequently, when we go over 1 unit to the right, the graph rises by 2 units.

Definitions A **linear function**, whose graph is a straight line, has the following form:

$$y = mx + b$$

The ***y*-intercept**, which is b, is the value of y when $x = 0$, or the place where the line crosses the y-axis. The **slope**, m, is the change in y over the change in x. Thus, if the line goes through points (x_1, y_1) and (x_2, y_2), the slope is as follows:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Quick Review Question 1

a. Execute the following R commands, which plot the above function, $f(t) = 2t + 1$, from $t = -3$ to 3:

```
t = seq(-3,3,.1)
plot(t, 2*t + 1, type="l")
```

b. By replacing xxxxxx with the appropriate equation, complete the command to plot f along with the equation of the line with the same slope as *f* but with *y*-intercept 3. Distinguish between the graphs of *f* and the new function, such as by color, line thickness, or dashing.

```
lines(t, xxxxxx, col="green", lty="dashed")
```

c. Add another function to the graph with a y-intercept of -3.
d. Describe the effect that changing the *y*-intercept has on the graph of the line.
e. Add another function to the graph with the same y-intercept, but a slope of -3.
f. Describe the effect that changing the slope has on the graph of the line.
g. Begin again with a fresh plot, this time with a slope of 1/4 and a y-intercept of -1. (Remember: the *plot*() function will start over with a fresh plot, whereas the lines() and points() functions add more information to an existing plot.)

Quadratic Function

In Module 2.2, "Unconstrained Growth and Decay," we considered a ball thrown upward off a bridge 11 m high with an initial velocity of 15 m/sec. The function of height of the ball with respect to time is the following quadratic function:

$s(t) = -4.9t^2 + 15t + 11$

The general form of a **quadratic function** is as follows:

$f(x) = a_2x^2 + a_1x + a_0$

where a_2, a_1, and a_0 are real numbers. As execution of the following commands show, the graph of the ball's height s(t) is a **parabola** that is concave down.

```
t = seq(-1,40.1);
plot(t,-4.9*t^2 + 15*t + 11)
```

The next two Quick Review Questions develop some of the characteristics of quadratic functions.

Definition A **quadratic function** has the following form:

$f(x) = a_2x^2 + a_1x + a_0$

where a_2, a_1, and a_0 are real numbers. Its graph is a parabola.

Quick Review Question 2

a. Execute the following R commands, which plot the above function, $s(t) = -4.9t^2 + 15t + 11$, from $t = -1$ to 4:

```
t = seq(-1,4,.1)
plot(t, -4.9*t^2 + 15*t + 11, type="l")
```

b. Give the commands to plot $s(t)$ and another function with the same shape that crosses the y-axis at 2. Have the graph of the new function be dashed.

c. Using calculus, determine the time t at which the ball reaches its highest point. Verify your answer by referring to the graph.

d. What effect does changing the sign of the coefficient of t^2 have on the graph?

Quick Review Question 3 In this question, we consider various transformations on a function.

a. Plot t^2, $t^2 + 3$, and $t^2 - 3$ on the same graph, using dashing (***lty***="dashed") for the second curve and dotting (***lty***="dotted") for the third.

b. Describe the effect of adding a positive number to a function.

c. Describe the effect of subtracting a positive number from a function.

d. Plot t^2, $(t + 3)^2$, and $(t - 3)^2$ on the same graph, using dashing for the second curve and dotting for the third.

e. Describe the effect of adding a positive number to the independent variable in a function.

f. Describe the effect of subtracting a positive number from the independent variable in a function.

g. Plot t^2 and $-t^2$ on the same graph, using dashing for the second curve.

h. Describe the effect of multiplying a function by -1.

i. Plot t^2, $5t^2$, and $0.2t^2$ on the same graph, using dashing for the second curve and dotting for the third.

j. Describe the effect of multiplying the function by number greater than 1.

k. Describe the effect of multiplying the function by positive number less than 1.

Polynomial Function

Linear and quadratic functions are polynomial functions of degree 1 and 2, respectively. The general form of a **polynomial function of degree *n*** is as follows:

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

where $a_n, \ldots, a_1$, and a_0 are real numbers and n is a nonnegative integer. The graph of such a function with degree greater than 1 consists of alternating hills and valleys. The quadratic function of degree 2 has one hill or valley. In general, a polynomial of degree n has at most $n - 1$ hills and valleys.

Definition A **polynomial function of degree *n*** has the following form:

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$

where $a_n, \ldots, a_1$, and a_0 are real numbers and n is a nonnegative integer.

Quick Review Question 4

a. Execute the following R commands that plot the polynomial function $p(t) = t^3 - 4t^2 - t + 4$ from $t = -2$ to 5:

```
t = seq(-2,5,0);
plot(t, t^3 - 4*t^2 - t + 4)
```

b. To what value does p(t) go as t goes to infinity?
c. To what value does $p(t)$ go as t goes to minus infinity?
d. Plot $p(t)$ and another function with each coefficient of t having the opposite sign as in $p(t)$. Have the new function be dashed.
e. To what does the new function from Part d go as t goes to infinity?
f. To what does the new function from Part d go as t goes to minus infinity?

Square Root Function

The square root function, whose graph the following commands generate, is increasing and concave down. Its domain and range are the set of nonnegative real numbers.

```
t = seq(0,12,0.1);
plot(t, sqrt(t))
```

Quick Review Question 5 Plot each of the following transformations of the square root function. The **square root function** in R is ***sqrt***().
a. Move the graph to the right 5 units.
b. Move the graph up 3 units.
c. Rotate the graph around the x-axis
d. Double the height of each point.

Exponential Function

In Module 2.2, "Unconstrained Growth and Decay," we considered situations where the rate of change of a quantity, such as the size of a population, is directly proportional to the size of the population, such as $dP/dt = 0.1P$ with initial population $P_0 = 100$. As we saw, the solution to this differential equation is the exponential function $P = 100e^{0.1t}$ whose graph is in Figure 3.2.3 of that module. Similarly, the solution to the differential equation $\frac{dQ}{dt} = -0.000120968Q$ for radioactive decay is $Q = Q_0e^{-0.000120968t}$ with graph in that module's Figure 2.2.4. As indicated in both examples, the coefficient is the initial amount and the coefficient of t is the continuous rate. For a positive rate, the function increases and is concave up; while a negative rate results in a decreasing, concave-up function.

The base can be any positive real number, not just e, which is approximately 2.71828. For example, we can express $P = 100e^{0.1t}$ as an exponential function with base 2. Setting $100(2^{rt})$ equal to $100e^{0.1t}$, we cancel the 100s, take the natural logarithm of both sides, and solve for r, as follows:

$100e^{0.1t} = 100(2^{rt})$
$0.1t = \ln(2^{rt})$
$0.1t = rt \ln(2)$
$r = 0.1/\ln(2) = 0.14427$, when $t \neq 0$

Thus, $P = 100e^{0.1t} = 100(2^{0.14427t})$.

Definition An **exponential function** has the following general form:

$P(t) = P_0 a^{rt}$

where P_0, a, and r are real numbers.

Quick Review Question 6 Complete the following questions about exponential functions. The exponential function with base e in R is ***exp***().

- **a.** Define an exponential function $u(t)$ with initial value 500 and continuous rate 12%.
- **b.** Plot this function.
- **c.** On the same graph, plot exponential functions with initial value 500 and continuous rates of 12%, 13%, and 14%. Which rises the fastest?
- **d.** Express the function $u(t)$ as an exponential function with base 4.

Quick Review Question 7

- **a.** Define an exponential function $v(t)$ with initial value 5 and continuous rate -82%.
- **b.** Plot this function.
- **c.** Plot $v(t)$ and $v(t) + 7$ on the same graph with the latter dashed.
- **d.** What effect does adding 7 have on the graph?
- **e.** As t goes to infinity, what does $v(t)$ approach?
- **f.** As t goes to infinity, what does $v(t) + 7$ approach?
- **g.** Copy the answer to Part b. In the copy, plot $v(t)$ and $-v(t)$.
- **h.** What effect does negation (multiplying by -1) have on the graph?
- **i.** Copy the answer to Part g. In the copy, plot $v(t)$ and $7 - v(t)$.
- **j.** As t goes to infinity, what does $7 - v(t)$ approach?
- **k.** Give the value of $7 - v(t)$ when $t = 0$.

Quick Review Question 8 Complete this question, which considers a function that has an independent variable t as a factor and as an exponent.

- **a.** Plot $12te^{-2t}$ from $t = 0$ to $t = 5$.
- **b.** Initially, with values of t close to 0, give the factor that has the most impact, t or e^{-2t}.
- **c.** As t gets larger, give the factor that has the most impact, t or e^{-2t}.

Logarithmic Functions

In Module 2.2 on "Unconstrained Growth and Decay," we employed the logarithmic function to obtain an analytical solution to the differential equation $dP/dt = 0.1P$ with initial population $P_0 = 100$. In that same module, the logarithmic function was useful in solving a problem to estimate the age of a mummy.

John Napier, a Scottish baron who considered mathematics a hobby, published his invention of logarithms in 1614. Unlike most other scientific achievements, his work was not built on that of others. His highly original invention was welcomed enthusiastically, because problems of multiplication and division could be reduced to much simpler problems of addition and subtraction using logarithms.

By definition, m is the **logarithm to the base 10**, or **common logarithm**, **of n** written as $\log_{10} n = m$ or $\log n = m$, provided m is the exponent of 10 such that 10m is n or

$\log_{10} n = m$ if and only if $n = 10^m$

A logarithm is an exponent, in this case, an exponent of 10. Thus,

$\log_{10} 1000 = 3$ because $1000 = 10^3$
$\log_{10} 1{,}000{,}000 = 6$ because $1{,}000{,}000 = 10^6$
$\log_{10} 0.01 = -2$ because $0.01 = 10^{-2}$

Because 10^m is always positive, we can only take the logarithm of positive numbers, so that the domain of a logarithmic function is the set of positive real numbers. However, the exponent m, which is the logarithm, can take on values that are positive, negative, or zero. Thus, the range of a logarithmic function is the set of all real numbers. The following commands generate the graph of the common logarithm.

```
x = seq(0,10,0.1);
plot(x, log10(x))
```

Because the logarithm is an exponent, the logarithmic function increases very slowly, and the graph is concave down.

In scientific applications, we frequently employ the **logarithm to the base e** or the **natural logarithm**. The notation for $\log_e n$ is **ln n**. Similarly to the common logarithm, we have the following equivalence:

$\ln n = m$ if and only if $n = e^m$

Moreover, the graph of the natural logarithm has a similar shape to that of the common logarithm generated above.

Definitions The **logarithm to the base b of n**, written **$\log_b n$**, is m if and only if b^m is n. That is, $\log_b n = m$ is equivalent to $n = b^m$. The **common logarithm** of n, usually written $\log n$, has base 10; and the **natural logarithm** of n, usually written **$\ln n$**, has base e.

In comparing the graph of $\ln x$ to that of x and $\sqrt{x}$ in Figure 8.2.6 in the text, we see that the linear and square root functions dominate the logarithmic function, which is in color.

Quick Review Question 9 In R, ***log***(x) is the natural logarithm, or logarithm to the base e, of x, which is written "ln x" in mathematics; ***log10***(x) is the common logarithm, or logarithm to the base 10, of x, which is "log x" in mathematics; and ***log2***(x) is the logarithm to the base 2 of x, which is "lg x" in mathematics.

a. Evaluate lg 8.

b. Write $y = \log 7$ as a corresponding equation involving an exponential function.

c. Evaluate $\ln(e^{5.3})$.

d. Evaluate $10^{\log(6.1)}$.

Logistic Function

In Module 2.3 on "Constrained Growth," we modeled the rate of change of a population with a carrying capacity that limited its size. The model incorporated the following differential equation with carrying capacity M, continuous growth rate r, and initial population P_0:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{M}\right)P$$

The resulting analytical solution, which is a **logistic function**, is as follows:

$$P(t) = \frac{MP_0}{(M - P_0)e^{-rt} + P_0}$$

Figure 2.3.1 of the "Constrained Growth" module depicts the characteristic *S*-curve of this function.

Quick Review Question 10 When plotting several functions together, use a solid line for the first, dashed for the second, and dotted for the third.

a. Plot the logistic function with initial population $P_0 = 20$, carrying capacity $M = 1000$, and instantaneous rate of change of births $r = 50\% = 0.5$ from $t = 0$ to 16 to obtain a graph as in Figure 2.3.1 of Module 2.3, "Constrained Growth."

b. On the same graph, plot three logistic functions that each have $M = 1000$ and $r = 0.5$ but P_0 values of 20, 100, and 200.

c. What effect does P_0 have on a logistic graph?

d. On the same graph, plot three logistic functions that each have $M = 1000$ and $P_0 = 20$ but r values of 0.2, 0.5, and 0.8.

e. What effect does r have on a logistic graph?

f. On the same graph, plot three logistic functions that each have $P_0 = 20$ and $r = 0.5$ but M values of 1000, 1300, and 2000.

g. What effect does M have on a logistic graph?

Trigonometric Functions

The sine and cosine functions are employed in many models where oscillations are involved. For example, projects in Module 4.2, "Predator-Prey Model," considered seasonal birth rates and fishing and employed the cosine and sine functions, respectively, to achieve periodicity.

To define the trigonometric functions sine, cosine, and tangent, we consider the point (x, y) on the unit circle of Figure 8.2.7 in the text. For the angle t off the positive x-axis, with t being positive in the counterclockwise direction and negative in the clockwise direction, the definitions of these trigonometric functions are as follows:

sin *t* = *y*
cos *t* = *x*
tan *t* = *y* / *x*

For example, if $x = 0.6$ and $y = 0.8$, then t is approximately 0.9273 radians, so that the following hold:

sin(0.9273) = 0.8
cos(0.9273) = 0.6

$\tan(0.9273) = 0.8 / 0.6 \approx 1.33$

For an angle of 0 radians, the opposite side, *y*, is zero, so that sin(0) = 0. An angle of π/2 results in (1, 0) being the point on the unit circle and the sine function achieving its maximum value of 1. The sine returns to 0 for the angle π = 180º. Then, sin(*t*) obtains its minimum, namely -1, at 3π/2, where the point on the unit circle is (0, -1). At *t* = 2π = 360º, the sine function starts cycling through the same values again. Figure 8.2.8 presents one cycle of the sine function, and Figure 8.2.9 gives a cycle of the cosine function.

Quick Review Question 11 This question concerns the sine function. In R, sin *t* is ***sin***(*t*).

a. Evaluate sin *t* where *x* = 0.6 and *y* = 0.8 for angle *t*.

b. Evaluate sin(π/3) where the corresponding point on unit circle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

(In R, the constant π is written as the two letters ***pi***.)

c. Give the domain of the sine function.

d. Give the range of the sine function.

e. Give the sine's **period**, or length of time before the function starts repeating.

f. Is sin *t* positive or negative for values of *t* in the first quadrant?

g. Is sin *t* positive or negative for values of *t* in the second quadrant?

h. Is sin *t* positive or negative for values of *t* in the third quadrant?

i. Is sin *t* positive or negative for values of *t* in the fourth quadrant?

Quick Review Question 12 This question concerns the cosine function. In *R*, cos *t* is ***cos***(t).

a. Evaluate cos(0).

b. Evaluate cos(π/2).

c. Evaluate cos(π).

d. Evaluate cos(3π/2).

e. Evaluate cos(π/3) where the corresponding point on unit circle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

f. Give the maximum value of cos *t*.

g. Give the minimum value of cos *t*.

h. Give the domain of the cosine function.

i. Give the period of the cosine function.

j. Is cos *t* positive or negative for values of *t* in the first quadrant?

k. Is cos *t* positive or negative for values of *t* in the second quadrant?

l. Is cos *t* positive or negative for values of *t* in the third quadrant?

m. Is cos *t* positive or negative for values of *t* in the fourth quadrant?

For a function of the form *f*(*t*) = *A* sin(*Bt*) or *g*(*t*) = *A* cos(*Bt*), where *A* and *B* are positive numbers, *A* is the **amplitude**, or maximum value of the function from the horizontal line going through the middle of the function. For example, *h*(*t*) = 2 sin(7*t*) has amplitude 2; the function oscillates between *y* values of -2 and 2. Because the period of the sine and cosine functions is 2π, the period of *f* and *g* above is 2π/*B*. When *t* = 0, *Bt* = 0. When *t* = 2π/*B*, *Bt* = *B*(2π/*B*) = 2π. Thus, the period of *h*(*t*) = 2 sin(7*t*) is 2π/7.

Quick Review Question 13 Plot each following pair of functions with the second function dashed.

- **a.** sin *t* and 2 sin(7*t*).
- **b.** sin *t* and a function involving sine that has amplitude 5 and period 6π.
- **c.** sin *t* and a function involving sine that has minimum value -2 and maximum value 4.
- **d.** sin *t* and a function involving sine that has amplitude 4 and crosses the t-axis at each of the following values of *t:* …, -π/6, π/3, 5π/6, ….
- **e.** cos *t* and a function involving cosine that has amplitude 3, period π, and maximum value 2 at *t* = π/5.
- **f.** sin(5t) and e^{-t}sin(5*t*). The latter is a function of decaying oscillations. The general form of such a function is Ae^{Ct}sin(*Bt*), where *A*, *B*, and *C* are constants.

The tangent function is also periodic. Because tan *t* = *y* / *x*, for a corresponding point (*x*, *y*) on the unit circle (see Figure 8.2.7), tan *t* = sin *t* / cos *t*. The graph of this function appears in Figure 8.2.10, and the next Quick Review Question explores some of its properties.

Quick Review Question 14 This question concerns the tangent function. In R, tan *t* is ***tan***(t).

- **a.** Evaluate tan(π/3) where the corresponding point on unit circle is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.
- **b.** Evaluate tan(0).
- **c.** Evaluate tan(π).
- **d.** Evaluate tan(π/2).
- **e.** As *t* approaches π/2 from values less than π/2, what does tan *t* approach?
- **f.** As *t* approaches π/2 from values greater than π/2, what does tan *t* approach?
- **g.** Evaluate tan(-π/2).
- **h.** As *t* approaches -π/2 from values less than -π/2, what does tan *t* approach?
- **i.** As *t* approaches -π/2 from values greater than -π/2, what does tan *t* approach?
- **j.** Give the range of the tangent function.
- **k.** Give all the values between -2π and 2π for which tan *t* is not defined.
- **l.** Give an angle in the third quadrant that has the same value of tan *t*, where *t* is in the first quadrant.
- **m.** Give an angle in the fourth quadrant that has the same value of tan *t*, where *t* is in the second quadrant.
- **n.** Give the period of the tangent function.