2.3 Rate of Change

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Introduction

This and the next module are for the student who has not had Calculus I or who wants a review of the material to prepare for the remainder of the course. The concept of rate of change from the current module is particularly important. The material in the next module is important but optional to understanding the remainder of the text. The required calculus background for the text is minimal, and students do not need to know formulas to understand the material or develop the models.

   Calculus is the mathematics of change. The concept of **instantaneous rate of change**, or **derivative**, from a first course in calculus is used throughout the text. For example, we consider the rates of change of populations of predators and prey to study the dynamics of their populations. The speed with which a radioactive substance decays certainly impacts the amount present at a particular time. In physics or as we drive a car, the rate of change of position with respect to time is the velocity.

   Differential calculus, which deals with problems involving the derivative, is one of the two major parts of calculus. The other is **integral calculus**, which we consider in the next module. These two modules consider the concepts of differential and integral calculus that we need to solve modeling problems.

Velocity

We begin the discussion of instantaneous rate of change by considering the average rate of change. Suppose on a windless day someone standing on a bridge holds a ball over the side and tosses the ball straight up into the air. After reaching its highest point, the ball falls, eventually landing in the water. The ball's height above the water (y) is a function (s) of time (t), so we write $y = s(t)$. Figure 2.3.1 gives a plot of a ball's height in meters versus time in seconds, while Table 2.3.1 lists some of the function's values. The graph is not a plot of the ball's trajectory, which is straight up and then straight down again. The height of the ball at water level is $y = 0$ m, and a negative value of y indicates that the ball is under water.

Figure 2.3.1   Height (y) in meters versus time (t) in seconds of a ball thrown straight up from a bridge
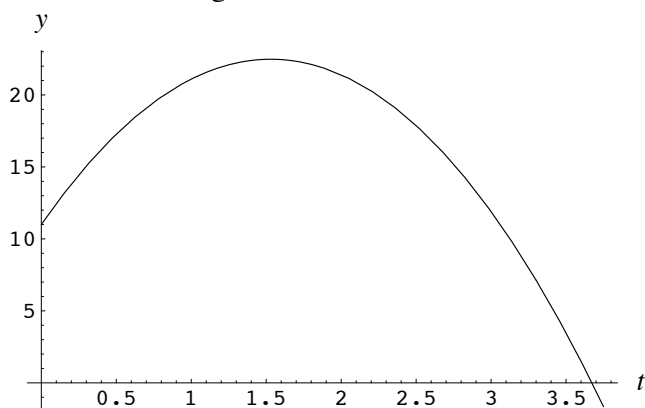
Table 2.3.1   Table of times and heights for a ball thrown straight up from a bridge

Time (t) in seconds	Height (y) in meters
0.00	11.0000
0.25	14.4438
0.50	17.2750
0.75	19.4938
1.00	21.1000
1.25	22.0938
1.50	22.4750
1.75	22.2438
2.00	21.4000
2.25	19.9437
2.50	17.8750
2.75	15.1937
3.00	11.9000
3.25	7.9938
3.50	3.4750
3.75	-1.6563

Quick Review Question 1   Consider the ball of Figure 2.3.1 and Table 2.3.1 in approximating the following, giving values and units:
a.   The height of the bridge
b.   The maximum height of the ball
c.   When the ball reaches its maximum height
d.   When the ball hits the water

As Figure 2.3.1 and Table 2.3.1 indicate, the ball changes position as time passes. The ball starts at one speed, moving up, and then slows down to a stop because of the effect of gravity. Subsequently, the ball starts to fall, picking up speed as it does, until finally splashes into the water,

We can approximate the velocity at any particular time t if we know the heights at times shortly before and after t and compute the average velocity over that time period. Thus, an understanding of average velocity is essential to that of instantaneous velocity, or instantaneous rate of change of position with respect to time. The **average velocity** is the ratio of the change in height, or position, to the change in time. For example, if at noon we are 60 km from home and at 2:00 p.m. 260 km from home, then over that time period, we drove at an average speed of $(260 - 60)/2 = 100$ km/hr. For the position function $s(t)$, the average velocity is the average rate of change of s with respect to t. From Table 2.3.1, we see that the average velocity in the first second, which is from time $a = 0$ sec to time $b = 1$ sec, is as follows:

$$\text{average velocity from 0 to 1 seconds} = \frac{s(1) - s(0)}{1.00 - 0.00} = \frac{21.1000 - 11.0000}{1.00} = 10.1 \text{ m/sec}$$

The units for average velocity, meters/second, are the units for the numerator, which are meters, over the units for the denominator, which are seconds.

> **Definition** Suppose $s(t)$ is the position of an object at time t, where $a \le t \le b$. The **average velocity**, or the **average rate of change of s with respect to t**, of the object from time a to time b is
> $$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(b) - s(a)}{b - a}$$

Quick Review Question 2   Determine the average velocity of the ball from Table 2.3.1
    a.   From $t = 1$ sec to $t = 2$ sec
    b.   From $t = 1$ sec to $t = 3$ sec

In the Quick Review Question 2, you should have determined the average velocity from 1 to 2 seconds to be 0.3 m/sec. Thus, on the average, the ball is moving faster the second before than the second after $t = 1$ sec. We can approximate the velocity of the ball at the instant $t = 1$ sec by finding the mean of the average velocities during the first second (10.1 m/sec) and the next second (0.3 m/sec), as follows:

$$\text{approximation of velocity at } t = 1 \text{ sec} = \frac{10.1 + 0.3}{2} = 5.2 \text{ m/sec}$$

Equivalently, we can evaluate the average velocity between times on either side of $t = 1$ sec. However, it is best to use known heights for times as close to $t = 1$ sec as possible, in this case for $t = 0.75$ sec and $t = 1.25$ sec.

Quick Review Question 3   Approximate the velocity of the ball at $t = 1$ sec by finding the average velocity from $a = 0.75$ sec to $b = 1.25$ sec.

A slight variation in the notation for determining the average velocity from a to b is advantageous for calculus. Instead of using b, we consider $b = a + \Delta t$, the initial time (a) plus a **change in time**, $\Delta t = b$ - a, with Δt pronounced delta-t (Δ is the fourth letter of the Greek alphabet). For example, if $a = 0.75$ and $b = 1.25$, then $\Delta t = 1.25$ - $0.75 = 0.50$ sec,

which is the change in time; and $b = a + \Delta t = 0.75 + 0.50 = 1.25$ sec. The following definition employs this notation.

Definition Suppose $s(t)$ is the position of an object at time t, where $a \le t \le b$. Then the **change in time**, Δt, is $\Delta t = b - a$; and the **change in position**, Δs, is $\Delta s = s(b) - s(a)$. Moreover, the **average velocity**, or the **average rate of change of s with respect to t**, of the object from time a to time $b = a + \Delta t$ is

$$\textbf{average velocity} = \frac{\textbf{change in position}}{\textbf{change in time}}$$

$$= \frac{\Delta s}{\Delta t}$$

$$= \frac{s(b) - s(a)}{b - a}$$

$$= \frac{s(a + \Delta t) - s(a)}{\Delta t}$$

Quick Review Question 4   Suppose we wish to determine the average velocity of the ball in Table 2.3.1 from time 2.25 sec to time 3.0 sec. Using the notation of the definition of average velocity involving Δt, determine the following, including units:

a. a
b. $s(a)$
c. Δt
d. $a + \Delta t$
e. $s(a + \Delta t)$
f. Δs
g.   The average velocity

To obtain the instantaneous velocity at $t = 1$ sec—that is, the *exact* velocity of the ball *precisely* one second after it starts to move, we determine the average velocity with changes in time, Δt, closer and closer to 0. Our answers approach a particular number, the instantaneous velocity at $t = 1$ sec. We say that we are taking the **limit of the average velocities as Δt approaches 0**, and we write the following:

$$\textbf{instantaneous velocity at 1 sec} = \lim_{\Delta t \to 0} \frac{s(1 + \Delta t) - s(1)}{\Delta t}$$

In the quotient, Δt can be positive or negative, but not zero.

Concept   Suppose that as x approaches some number c, $f(x)$ approaches a number L. We say the **limit of $f(x)$ as x approaches c is L**, and we write

$$\lim_{x \to c} f(x) = L$$

Definition   The **instantaneous velocity**, or the **instantaneous rate of change of** *s*
with respect to *t*, at $t = a$ is

$$\text{instantaneous velocity at } a \text{ sec} = \lim_{\Delta t \to 0} \frac{s(a + \Delta t) - s(a)}{\Delta t}$$

the limit of the average velocity from $t = a$ to $t = a + \Delta t$ as Δt approaches 0,
provided the limit exists.

Table 2.3.2 gives additional values of $s(t)$, possibly obtained experimentally, for t
close to 1 along with the average velocities between each such time and $t = 1$. The table
is in two parts. The left side has values of Δt starting at 0.10 and decreasing to 0.01 along
with columns for the corresponding $s(1 + \Delta t)$ and the *average* velocity, $\frac{[s(1 + \Delta t) - s(1)]}{\Delta t}$.
The right side of the table has negative values of Δt from -0.10 to a value closer to 0,
namely -0.01, along with the same second and third columns as on the left. Observing
the third columns for both sides, the average velocities appear to be converging to 5.20.
In fact,

$$\lim_{\Delta t \to 0} \frac{s(1 + \Delta t) - s(1)}{\Delta t} = 5.20$$

so that the instantaneous velocity of this ball at $t = 1$ sec is 5.20 m/sec.

Table 2.3.2   Average velocities between $(1, s(1)) = (1, 21.1)$ and $(1 + \Delta t, s(1 + \Delta t))$

Δt	$s(1 + \Delta t)$	$\dfrac{s(1 + \Delta t) - s(1)}{\Delta t}$	Δt	$s(1 + \Delta t)$	$\dfrac{s(1 + \Delta t) - s(1)}{\Delta t}$
0.10	21.571	**4.710**	-0.10	20.531	**5.690**
0.09	21.528	**4.759**	-0.09	20.592	**5.641**
0.08	21.485	**4.808**	-0.08	20.653	**5.592**
0.07	21.440	**4.857**	-0.07	20.712	**5.543**
0.06	21.394	**4.906**	-0.06	20.770	**5.494**
0.05	21.348	**4.955**	-0.05	20.828	**5.445**
0.04	21.300	**5.004**	-0.04	20.884	**5.396**
0.03	21.252	**5.053**	-0.03	20.940	**5.347**
0.02	21.202	**5.102**	-0.02	20.994	**5.298**
0.01	21.152	**5.151**	-0.01	21.048	**5.249**

Quick Review Question 5   Using Table 2.3.3 of values, estimate $\displaystyle\lim_{\Delta t \to 0} \frac{s(2 + \Delta t) - s(2)}{\Delta t}$
to one decimal place.

Table 2.33   Table for Quick Review Question 5

Δt	$\dfrac{s(2+\Delta t)-s(2)}{\Delta t}$	Δt	$\dfrac{s(2+\Delta t)-s(2)}{\Delta t}$
0.0005	27.8072	-0.0005	27.7928
0.0004	27.8058	-0.0004	27.7942
0.0003	27.8043	-0.0003	27.7957
0.0002	27.8029	-0.0002	27.7971
0.0001	27.8014	-0.0001	27.7986

Derivative

The limit $\displaystyle\lim_{\Delta t \to 0} \frac{s(a+\Delta t)-s(a)}{\Delta t}$ above has far more applications than instantaneous velocity at $t = a$. Because of its vast importance, the formula has a special name. In general, when the limit exists, $\displaystyle\lim_{\Delta t \to 0} \frac{s(a+\Delta t)-s(a)}{\Delta t}$ is the **derivative** of s with respect to t at a. We use two notations for the derivative of $y = s(t)$ with respect to t—the first is $\dfrac{dy}{dt}$, and the second is $s'(t)$. For the derivative of s at $t = 1$, which in this case is 5.20, we write $\dfrac{dy}{dt}\Big|_{t=1}$ or $s'(1) = 5.20$ m/sec. Notice that the units for the instantaneous and average velocities are the same, m/sec.

Definition   The **derivative of $y = s(t)$ with respect to t** at $t = a$ is the instantaneous rate of change of s with respect to t at a:

$$s'(a) = \frac{dy}{dt}\Big|_{t=a} = \lim_{\Delta t \to 0} \frac{s(a+\Delta t)-s(a)}{\Delta t}$$

provided the limit exists. If the derivative of s exists at a, we say the function is **differentiable** at a.

Quick Review Question 6   Suppose the population P of a colony of bacteria in millions is a function of time t in hours. Give the units of dP/dt.

        Although we will not do so here, we could show for the ball example that the derivative of s at $t = 0$ sec is $s'(0) = 15$ m/sec. Thus, initially, when the ball is at height $s(0) = 11.0$ m (see Table 2.3.1), the ball is increasing its height at a rate of 15 m/sec. With this information, we can estimate the height of the ball one second later at $t = 1$ sec as $11 + 15 = 26$ m. Because of the pull of gravity, the ball does not get quite that high; but the derivative can help us make estimates for the future as well as understand the current situation.

**Quick Review Question 7**   We know from Table 2.3.1 that $s(2.5) = 17.875$ m. Suppose $s'(2.5) = -9.5$.
   a.   Give the units of 2.5.
   b.   Give the units of -9.5.

c.   Using this information, estimate $s(3.5)$. Include units.
**d.**   Interpret this information.

Slope of Tangent Line

In this section, we consider graphically the instantaneous velocity, or the derivative $s'(t)$. Suppose we again wish to examine the velocity of the ball at time $t = 1$ sec. If we keep zooming in on the graph of the height of the ball $y = s(t)$ at time $t = 1$ sec, we observe an interesting phenomenon—The appearance of these graphs as we zoom in is increasingly linear. Figures 2.3.2 through 2.3.4 show plots of y versus t as we zoom in, first from $t = 0.75$ to 1.25 sec, then from $t = 0.9$ to 1.1 sec, and finally from $t = 0.99$ to 1.01 sec. At close range in Figure 2.3.4, the tangent line to $y = s(t)$ at $t = 1$ approximates the graph of $y = s(t)$.

Figure 2.3.2   From Figure 2.3.1, graph of $y = s(t)$ from $t = 0.75$ to 1.25 sec
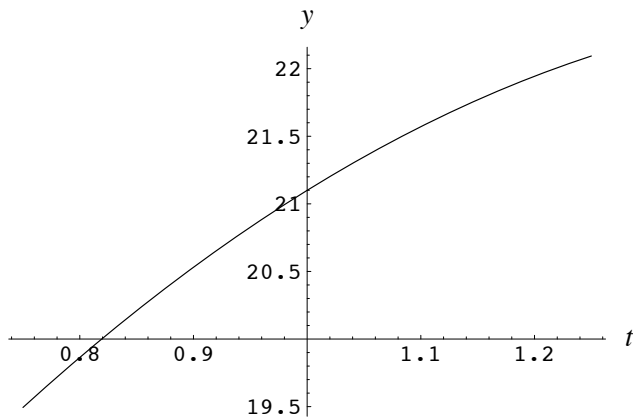
Figure 2.3.3   From Figure 2.3.1, graph of $y = s(t)$ from $t = 0.9$ to 1.1 sec
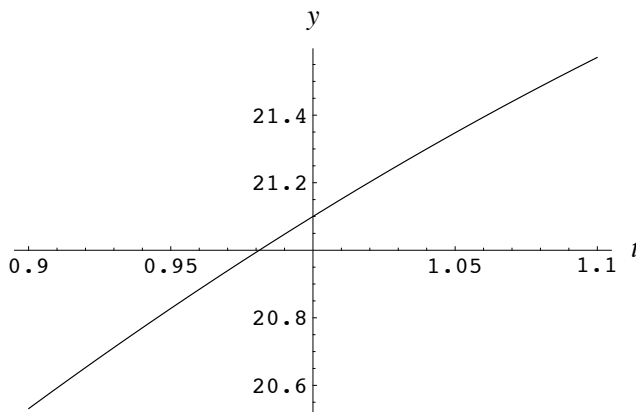
Figure 2.3.4   From Figure 2.3.1, graph of $y = s(t)$ from $t = 0.99$ to 1.11 sec
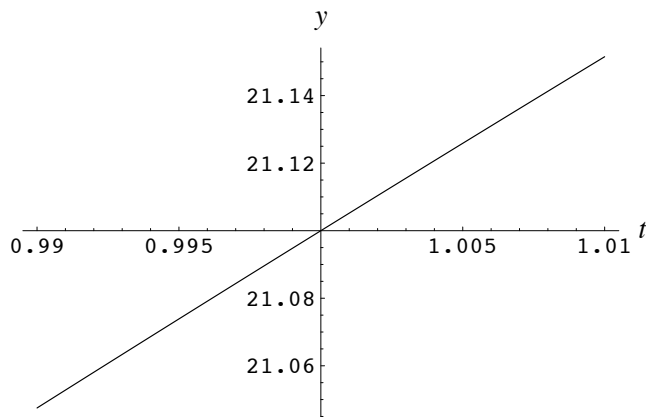
In general, the slope of the tangent line to a curve at a point is the derivative of the function at that point. Figures 2.3.5 through 2.3.7 illustrate that the secant line through t = 1 and $t = 1 + \Delta t$ approaches the tangent line as Δt gets smaller. In Figure 2.3.5, the change in time is $\Delta t = 1.25$, and the slope of the secant line is as follows:

$$\frac{19.9437 - 21.1}{1.25} = \text{-}0.925$$

As the next Quick Review Question shows, the slopes for $\Delta t = 0.75$ and 0.25 are 1.525 and 3.975, respectively (see Figures 2.3.6 and 2.3.7).

Definition   The **slope** of a non-vertical line through two distinct points (x_1, y_1) and (x_2, y_2) is $(y_2 - y_1) / (x_2 - x_1)$.

Quick Review Question 8
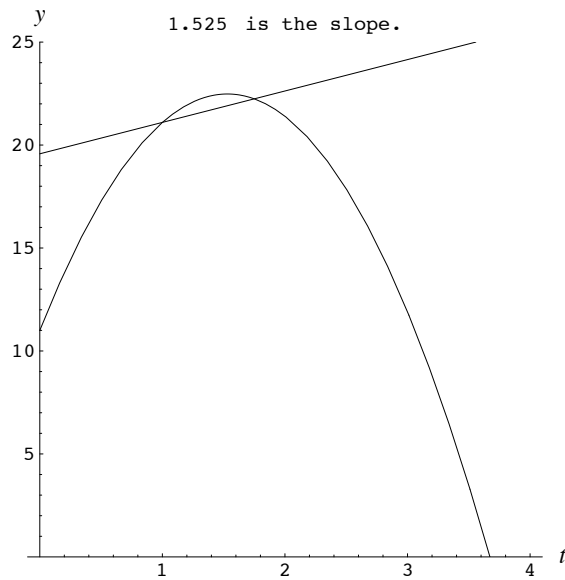   a.   Show the calculation of the slope of the secant line through $(1, 21.1)$ and $(1.75, 22.2428)$ for Figure 2.3.6.
   b.   Show the calculation of the slope of the secant line through $(1, 21.1)$ and $(1.25, 22.0938)$ for Figure 2.3.7.
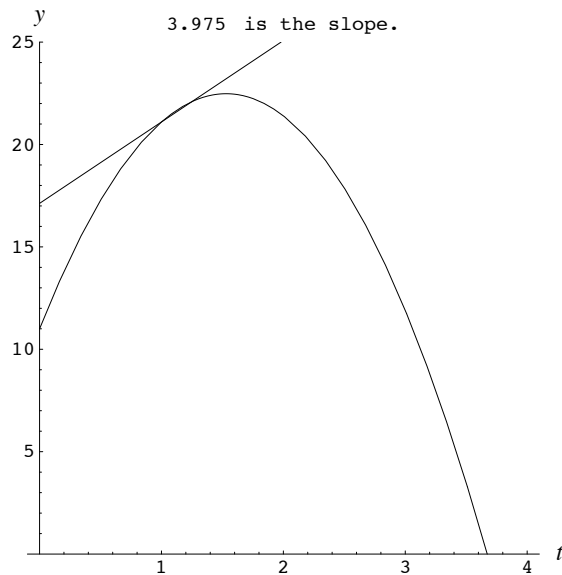
Figure 2.3.5   Secant line through $(1, 21.1)$ and $(2.25, 19.9437)$ with $\Delta t = 1.25$ sec and
slope of -0.925 m/sec
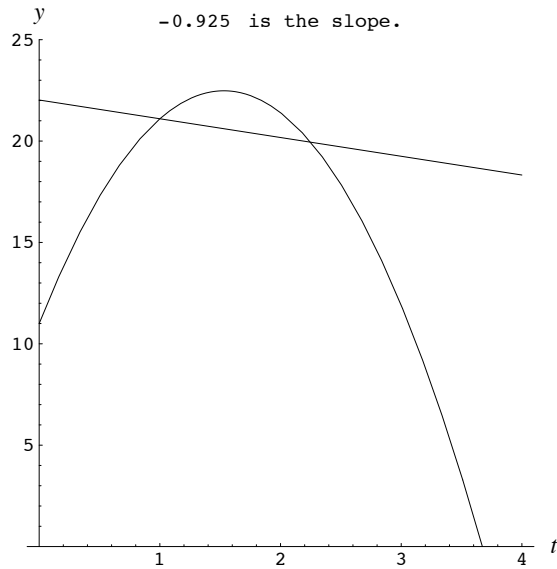
Figure 2.3.6   Secant line through $(1, 21.1)$ and $(1,75, 22.2438)$ with $\Delta t = 0.75$ sec and
slope of 1.525 m/sec

Figure 2.3.7 Secant line through $(1, 21.1)$ and $(1.25, 22.0938)$ with $\Delta t = 0.25$ sec and slope of 3.975 m/sec

As Figures 2.3.5 through 2.3.7 along with the graph of the tangent line in Figure 2.3.8 illustrate, the secant lines approach the tangent line as Δt goes to 0. Thus, the slopes of these secant lines approach the slope of the tangent line at $t = 1$. The slopes of the secant lines through $(1, 21.1)$ and $(1 + \Delta t, s(1 + \Delta t))$ are average velocities, and the limit as Δt approaches 0 is the instantaneous velocity, or the derivative, of the function s at 1. Thus, we can interpret the derivative at a point as the slope of the tangent line to the curve at that point. We also call this slope the **slope of the curve** at that point.

Concept Geometrically, the **derivative** at a point is the slope of the tangent line to the curve at that point.

Figure 2.3.8 Tangent line to curve at $(1, 21.1)$ and slope of 5.2 m/sec
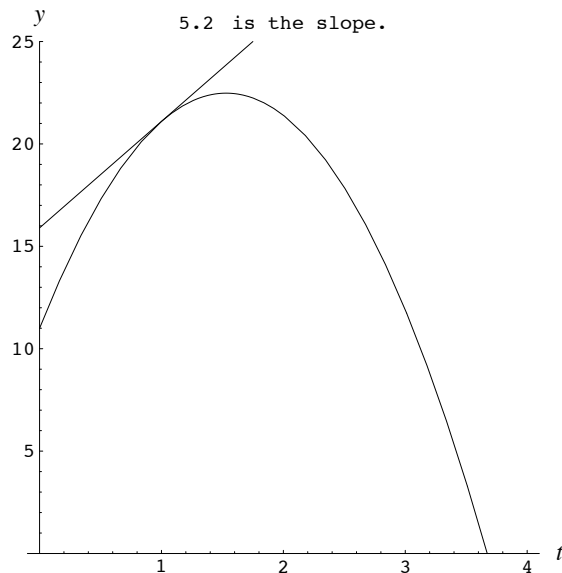
The derivative of s at 1 is a number, 5.20, which is the slope of the tangent line to the curve s at $t = 1$ (see Figure 2.3.8). However, the tangent lines, and consequently their slopes, depend on the point on the curve. Table 2.3.4 presents a list of the slopes of some of the tangent lines to the curve s. As expected for this graph, the slopes are positive where the curve is increasing on the left and negative where the curve is decreasing on the right. Because the slope of the tangent line to the curve, and consequently the derivative, depends on t, we can define a **derivative function**, as follows:

$$s'(t) = \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}, \quad \text{provided the limit exists}$$

Table 2.3.4   List of some values of t and the slopes of the tangent lines to s of Figure
           2.3.8 at t

t	Slope of Tangent Line at t
0.0	15.0
0.5	10.1
1.0	5.2
1.5	0.3
2.0	-4.6
2.5	-9.5
3.0	-14.4
3.5	-19.3

Definition The **derivative function of $y = s(t)$ with respect to t** is the instantaneous rate
     of change of s, provided the limit exists:

$$\frac{dy}{dt} = s'(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

Quick Review Question 9   Use Table 2.3.4 to evaluate the derivative function at the
     requested values.
     a. $s'(0.5)$ **b.** $s'(3.0)$

Although we will not verify the result, it can be shown that the derivative function for $y = s(t)$ is $s'(t) = -9.8t + 15$. As we have justified, $s'(1) = 5.20$, which is $-9.8(1) + 15$.

Quick Review Question 10   Using the fact that $s'(t) = -9.8t + 15$, determine the
     following along with their units:
     a. $s'(1.3)$
     b.   The slope of the tangent line to s at $t = 2.9$ sec
     c.   The instantaneous rate of change of s at $t = 0.4$ sec

Differential Equations

A **differential equation** is an equation that contains a derivative. For example, if y is a position of a ball above water at time t, then the rate of change, or derivative, of y with

respect to t is the velocity. Suppose the velocity function is $v(t) = dy/dt = s'(t) = -9.8t + 15$, and the initial position, or initial condition, is $y_0 = s(0) = 11$. Thus, we have the following differential equation with initial condition:

$dy/dt = -9.8t + 15$ and $y_0 = 11$

or

$s'(t) = -9.8t + 15$ and $s(0) = 11$

To solve the differential equation means to find a function $y = s(t)$ that satisfies the differential equation and initial condition(s). As we discuss in the next module, $y = s(t) = -4.9t^2 + 15\,t + 11$ is the solution to the above differential equation. Verifying the solution, we take the derivative of $y = s(t)$ to obtain $dy/dt = s'(t) = -9.8t + 15$. Moreover, substituting 0 for t in $s(t)$, we find that $y_0 = s(0) = 11$ also holds. The function $y = s(t) = -4.9t^2 + 15\,t + 11$ gives the height above the water as a function of time for the ball example of this module. We can obtain the height values in Table 2.3.1 and Table 2.3.2 by substituting appropriate values of t into the function.

Definitions   A **differential equation** is an equation that contains one or more derivatives. An **initial condition** is the value of the dependent variable when the independent variable is zero. A **solution** to a differential equation is a function that satisfies the equation and initial condition(s).

Quick Review Question 11 It can be shown that the derivative of $y = 3t^6 + 7$ is $18t^5$. Why is $y = 3t^6 + 7$ not a solution to the differential equation $dy/dt = 18t^5$ with initial condition $y_0 = 14$?

Second Derivative

Acceleration is the rate of change of velocity with respect to time, and an instantaneous rate of change is a derivative. Thus, the derivative of a velocity function, $v(t)$, is an acceleration function, $a(t) = v'(t)$. However, a velocity function itself is a derivative, the derivative of a position function with respect to time; for $y = s(t)$, $v(t) = s'(t) = dy/dt$. Consequently, acceleration is the derivative of the derivative of position. If we take the derivative of a position function and then the derivative of the result, we obtain the corresponding acceleration function. We say that we have taken the **second derivative** of the position function and write $a(t) = s''(t) = \dfrac{d^2y}{dt^2}$. Notice the placements of the 2's in the latter stacked notation. This notation elicits the units for the second derivative. If velocity is in m/sec, then the units for acceleration are (m/sec)/sec or, inverting and multiplying, m/sec^2.

Definition   **Acceleration** is the rate of change of velocity with respect to time.

Definition   The **second derivative** of a function $y = s(t)$ is the derivative of the derivative of y with respect to the independent variable t. The notation for this second derivative is $s''(t)$ or $\dfrac{d^2y}{dt^2}$.

Quick Review Question 12
    a.   Suppose $z = f(x) = x^3$; $h(x) = f'(x) = 3x^2$; and $g(x) = h'(x) = 6x$. Evaluate $f''(x)$.
    b.   Give another notation for $f''(x)$.
    c.   If velocity is in ft/sec, give the units for acceleration.

Exercises

1.   Use the following table of positions (s) of a car at various times (t).

t (hr)	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0
s (km)	43.2	31.7	22.3	16.5	15.1	18.5	26.1	36.6	48.5	59.8	68.8

    a.   Give the average velocity with units of the car between $t = 5.0$ hr and 9.0 hr.
    b.   Estimate the velocity with units of the car at $t = 6.5$ hr.
    c.   Estimate the rate of change with units of the car at $t = 4.5$ hr.

2.   For the graph in Figure 2.3.9, estimate the following:
    a.   The average rate of change of the function from $x = 0.5$ to $x = 1.5$
    b.   The average rate of change of the function from $x = 1.5$ to $x = 2$
    c.   The slope of the tangent line to the function at $x = 1$
    d.   The instantaneous rate of change of the function at $x = 1$
    e.   The derivative of the function at $x = 1$
    f.   The slope of the tangent line to the function at $x = 0.5$
    g.   The instantaneous rate of change of the function at $x = 0.5$
    h.   The derivative of the function at $x = 0.5$

Figure 2.3.9   Graph for Exercise 2
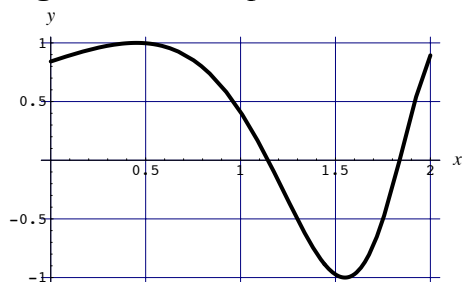
3.   Table 2.3.5 shows values for $\dfrac{f(2+\Delta x) - f(2)}{\Delta x}$ as Δx approaches 0 through positive values on the top part of the table and through negative values on the bottom part of the table. Estimate the following to two decimal places:
    a. $\lim\limits_{\Delta x \to 0} \dfrac{f(2+\Delta x) - f(2)}{\Delta x}$
    b. $f'(2)$
    c. $\left.\dfrac{dy}{dx}\right|_{x=2}$, where $y = f(x)$
    d.   The slope of the tangent line to the graph of f at 2
    e.   The instantaneous rate of change of f at 2

Table 2.3.5   Table for the Exercise 3

Δx	0.0051	0.0041	0.0031	0.0021	0.0011	0.0001
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x}$	3.1955	3.2187	3.2419	3.2649	3.2878	3.3107

Δx	-0.0051	-0.0041	-0.0031	-0.0021	-0.0011	-0.0001
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x}$	3.4275	3.4053	3.3829	3.3605	3.3379	3.3152

4. Suppose $N = f(t)$ is the number of atoms of radium-226 at time t, which is in days. Because radium-226 is radioactive, the substance is decaying.
   a. Give the units for $f'(t)$.
   b. Give the sign for $f'(t)$.
   c. Give the units for $f''(t)$.

5. Suppose the number of tuna $y = T(t)$ in the Mediterranean Sea is a function of time t in years since 1914.
   a. Give the units of the rate of change of tuna numbers with respect to time.
   b. Give two notations for the rate of change of tuna numbers in 1918.
   c. Is it desirable for this rate to be positive or negative?
   d. Give the units for the second derivative of y.
   e. Give two notations for the second derivative of y in 1918.

6. Suppose the time of a chemical reaction, T (in minutes), to oxidize an alcohol is a function of the amount of a catalyst (alcohol dehydrogenase), a (in milliliters), that is present. Thus, $T = f(a)$.
   a. If $f(4) = 13$, give the units of 4 and 13.
   b. If $f'(4) = -2$, give the units of 4 and -2.
   c. Interpret these statements taken together.

7. Suppose on Day 4 of an epidemic in a school that the rate at which students are developing influenza is 25 students/day.
   a. Give an interpretation of this rate as the derivative of a function.
   b. If 263 students have influenza on Day 4, estimate the number of students who will have influenza on Day 5.

8. $T = f(t)$ is the temperature in degrees Celsius of a beaker at time t (in hours) after someone places the beaker in a refrigerator.
   a. Give the units for $f'(t)$.
   b. Give the sign for $f'(t)$.
   c. Give the units for $f''(t)$.

9. The size of a drug's dose, S (in milligrams), depends on the weight of the patient, w (in pounds), so that $S = f(w)$.
   a. Interpret $f(150) = 200$.
   b. Interpret $f'(150) = 6$.
   c. Using the information from Parts a and b, estimate $f(151)$.
   d. Using the information from Parts a and b, estimate $f(155)$.

10. Use a computational tool or calculus to solve the following differential equation:
$$dP/dt = 0.3P - 20, P_0 = 35$$

Project

1. Using a computational tool, such as *Maple*, *Mathematica*, or *MATLAB*, develop a file to explain and illustrate the material of this module. Use different functions than appear in the module for your examples. Employ looping and printing to generate sequences of values as in Tables 2.3.1and 2.3.2.

Answers to Quick Review Questions

1.   **a.**   About 11 m
      b.   About 22.5 m
      c.   About 1.5 sec
      d.   About 3.7 sec

2.   **a.**   Average velocity from 1 to 2 seconds $= \dfrac{s(2)-s(1)}{2-1} = \dfrac{21.4-21.1}{1} = 0.3$ m/sec
      b.   Average velocity from 1 to 3 seconds $= \dfrac{s(3)-s(1)}{3-1} = \dfrac{11.9-21.1}{2} = $ -4.6 m/sec

3.   Approximation of velocity at $t = 1$ sec $= \dfrac{s(1.25)-s(0.75)}{1.25-0.75} = \dfrac{22.0938-19.4938}{0.5} = $
   5.2 m/sec, which in this problem is the same as the mean of the average velocities during the first and second seconds. These values do not necessarily always agree.

4.   **a.**   2.25 sec
      b.   19.9437 m
      c.   0.75 sec.
      d.   3.0 sec.
      e.   11.9 m
      f.   11.9000 - 19.9437 = -8.0437 m
      g.   -8.0437/0.75 = -10.725 m/sec. With up being positive, the average velocity is negative because the ball is falling during this time period.

5.   27.8
6.   millions of bacteria/hour
7.   **a.**   sec
      b.   m/sec
      c.   17.875 + -9.5 = 8.375 m
      d.   At time 2.5 sec, after one additional second (at time 3.5 sec), we estimate that the height of the ball will be 8.375 m.

8.   **a.** $\dfrac{21.1-22.2438}{-0.75} = 1.525$
      b. $\dfrac{21.1-22.0938}{-0.25} = 3.975$

9.   **a.**   10.1
      b.   -14.4

10. **a.** $s'(1.3) = $ -9.8(1.3) + 15 = 2.26 m/sec
      b. $s'(2.9) = $ -9.8(2.9) + 15 = -13.42 m/sec
      c. $s'(0.4) = $ -9.8(0.4) + 15 = 11.08 m/sec

11. Substituting 0 for t, $y = 7$, not 14.
12. **a.** $6x$

b. $\dfrac{d^2z}{dx^2}$

c.   ft/sec^2

Reference

Hughes-Hallet, Deborah, Andrew M. Gleason, William G. McCallum, et al. 2004.
  Single Variable Calculus. 3rd ed. New York, NY: John Wiley & Sons.