2.4 Fundamental Concepts of Integral Calculus

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By Angela B. Shiflet and George W. Shiflet
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Introduction

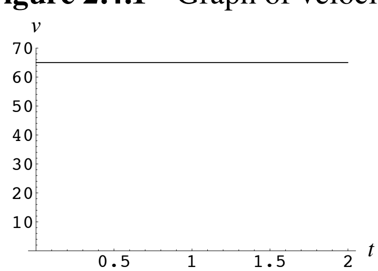
In the last module, we examined some of the fundamental concepts of differential calculus, while in this module we consider several of the high points of the other major part of calculus, **integral calculus**. In a sense, **integration**, or determining the integral of a function, is the reverse of differentiation, or finding the derivative. For example, we can integrate the rate of change function for the amount of a radioactive substance to determine the total amount of decay over a period of time. As we discuss in this module, we can integrate the velocity function for a ball tossed in the air from a bridge—that is, we can integrate the rate of change function for distance of a ball above the water as in Module 2.3 on "Rate of Change"—to obtain the total distance covered from one time to another.

The material in this module is needed for some elective sections in the remainder of the text. While the concept of rate of change from the previous module is essential to understanding the remainder of the text, the concepts of the current module, though extremely important for higher-level computational science, are optional for this introduction to the subject.

Total Distance Traveled and Area

Suppose a car travels along a straight expressway for 2 hours at a constant velocity of 65 km/hr. How far does the car go in that time? Clearly, the answer is (2 hr)(65 km/hr) = 130 km. Figure 2.4.1 gives the graph of this velocity function $v = f(t) = 65$. Notice that the total distance traveled is the area under this curve from $t = 0$ hr to $t = 2$ hr.

Figure 2.4.1 Graph of velocity function $v(t) = 65$ km/hr with t in hr



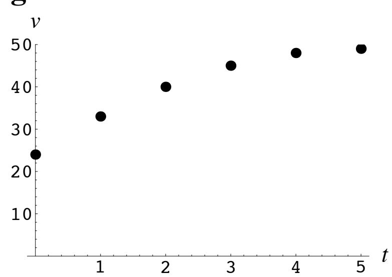
Quick Review Question 1 Suppose someone sets the cruise control to drive at 70 km/hr for half-an-hour. Then, the speed limit changes, and the person resets the cruise control for 60 km/hr for the next hour-and-a-half. Approximate the total distance traveled.

As another example, suppose a racecar moves with increasing velocity. The velocities in m/sec are recorded at various times, measured in seconds, in Table 2.4.1 and are displayed in Figure 2.4.2. We can estimate the total distance traveled over the 5-second period in several ways, including calculating under- and over estimates.

Table 2.4.1 Values for the velocity of a car at certain times

t (sec)	0	1	2	3	4	5
v (m/sec)	24	33	40	45	48	49

Figure 2.4.2 Plot of velocities from Table 2.4.1 with t in sec and v in m/sec

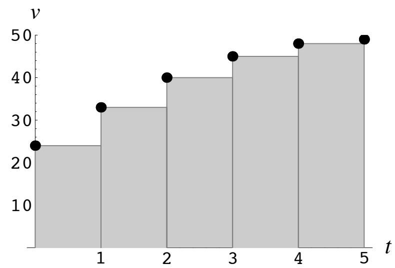


One underestimate of the total distance traveled takes the lowest velocity in each 1-second interval. For example, during the first second, from $t = 0$ sec to $t = 1$ sec, the lowest velocity (24 m/sec) occurs initially. If we travel at a constant velocity of 24 m/sec for 1 sec, then during that second we cover a total distance of (24 m/sec) (1 sec) = 24 m. As Figure 2.4.3 illustrates, this result is also the area of the rectangle in the first interval from the t-axis to the leftmost point (0, 24). This rectangle has height 24 and width 1. Because the velocity is increasing, during the next and each subsequent 1-second interval, the minimum velocity also occurs on the left. Summing the underestimates of the distances traveled for the five intervals, we obtain an underestimate of the total distance traveled for the 5 seconds, as follows:

underestimate = (24)(1) + (33)(1) + (40)(1) + (45)(1) + (48)(1) = 190 m

Although six points appear in Table 2.4.1 and Figure 2.4.2, for the computation, we use only five points, one for each interval. The estimate, 190 m, for the total distance traveled is also the area of the shaded rectangles in Figure 2.4.3.

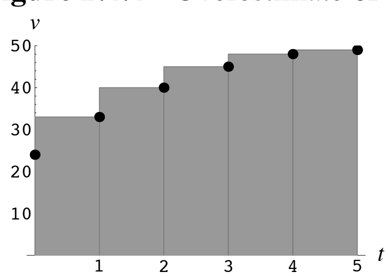
Figure 2.4.3 Underestimate of total distance traveled in m using intervals of 1 sec



We can obtain an overestimate of the total distance traveled by using the largest velocity in each of the five intervals. For this increasing function, these velocities occur on the right of each interval. Consequently, we compute the sum of the areas of the rectangles that touch the intervals' rightmost points (see Figure 2.4.4) to obtain an overestimate of the total distance traveled:

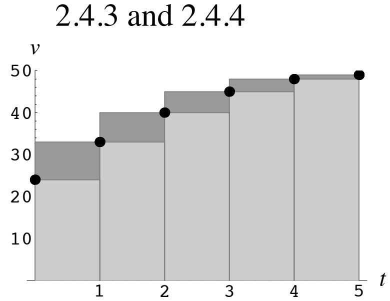
overestimate = (33)(1) + (40)(1) + (45)(1) + (48)(1) + (49)(1) = 215 m

Figure 2.4.4 Overestimate of total distance traveled in m using intervals of 1 sec



The actual distance is between the two estimates, 190 m and 215 m. Figure 2.4.5 illustrates these estimates and shows the difference (25 m) between the two with darker shading.

Figure 2.4.5 Over- and underestimates of the total distance in m traveled from Figures 2.4.3 and 2.4.4



Quick Review Question 2 Suppose after an hour of traveling, a bicyclist starts riding up a mountain with decreasing velocity, according to Table 2.4.2:

Table 2.4.2 Table for Quick Review Question 2

t (hr)	1.0	1.5	2.0	2.5	3.0

v (km/hr)	80.5	68.1	44.9	30.1	25.1

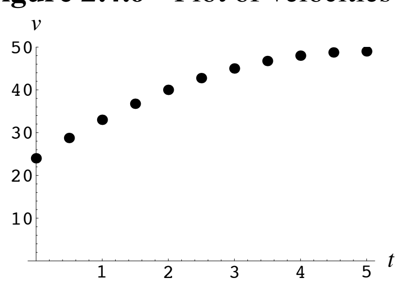
a. Using intervals of a half-an-hour, determine the best underestimate of the total distance the bicyclist travels up the mountain.
b. Repeat Part a to obtain an overestimate.

We can obtain a better estimate by using more frequent velocity measurements. Table 2.4.3 and Figure 2.4.6 give the velocities in half-second intervals.

Table 2.4.3 Additional values for the velocity of a car at certain times

t (sec)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
v (m/sec)	24.00	28.75	33.00	36.75	40.00	42.75	45.00	46.75	48.00	48.75	49.00

Figure 2.4.6 Plot of velocities (m/sec) versus time (sec) from Table 2.4.3



As we did for intervals of width 1 for the underestimate, we employ the minimum velocity in each interval as if the car were traveling at that velocity throughout that small time period. For example, in the first interval, a car traveling at 24.00 m/sec for 0.5 sec covers (24.00 m/sec)(0.5 sec) = 12.00 m during that half second. We must be careful to multiply by the change in time, 0.5 sec; the car does not move 24.00 m during the half-second interval but only half that amount. As above, the estimate of the distance is the area of the rectangle for the width of the interval to the point. Figure 2.4.7 shades the 10 rectangles of width 0.5 whose total area is, as follows:

underestimate = 24.00(0.5) + 28.75(0.5) + 33.00(0.5) + 36.75(0.5) + 40.00(0.5) + 42.75(0.5) + 45.00(0.5) + 46.75(0.5) + 48.00(0.5) + 48.75(0.5) m
= 196.875 m

This area is an underestimate of the total distance traveled, 196.875 m. To minimize the number of multiplications, we can factor out 0.5, adding the velocities and then multiplying by the length of the interval, as follows:

underestimate = (24.00 + 28.75 + 33.00 + 36.75 + 40.00 +
42.75 + 45.00 + 46.75 + 48.00 + 48.75)(0.5) m
= 196.875 m

Figure 2.4.8 shades the corresponding overestimate with the following computation:

overestimate = 28.75(0.5) + 33.00(0.5) + 36.75(0.5) + 40.00(0.5) +
42.75(0.5) + 45.00(0.5) + 46.75(0.5) + 48.00(0.5) +

$$
\begin{aligned}
& \quad 48.75(0.5) + 49.00(0.5) \\
= & \quad (28.75 + 33.00 + 36.75 + 40.00 + 42.75 + \\
& \quad 45.00 + 46.75 + 48.00 + 48.75 + 49.00)\,(0.5) \\
= & \quad 209.375 \text{ m}
\end{aligned}
$$

Figure 2.4.9 indicates the difference between these two estimates, 209.375 m - 196.875 m = 12.5 m, which is one-half the difference for intervals of length 1. The estimates are converging as the width of an interval goes to zero and, simultaneously, the number of intervals goes to infinity.

Figure 2.4.7 Underestimate of total distance traveled in m using intervals of 0.5 sec

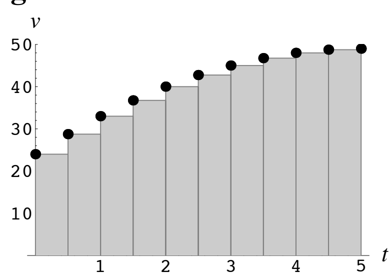


Figure 2.4.8 Overestimate of total distance traveled in m using intervals of 0.5 second

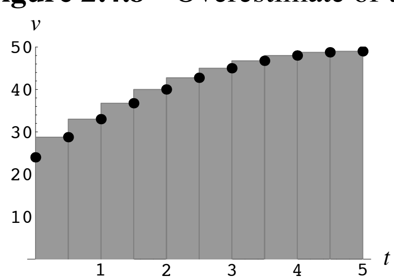


Figure 2.4.9 Over- and underestimates of the total distance in m traveled from Figures 2.4.7 and 2.4.8

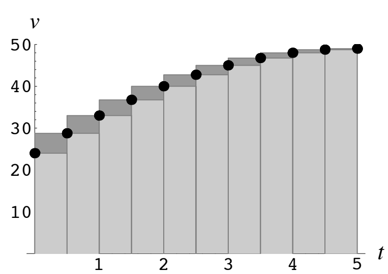


Table 2.4.3 gives the values of Table 2.4.2 along with notations. The times for the velocities are $t_0 = 0.0$, $t_1 = 0.5$, $t_2 = 1.0, \ldots, t_{10} = 5.0$, and the corresponding velocities are $f(t_0) = f(0.0) = 24.00, f(t_1) = f(0.5) = 28.75, f(t_2) = f(1.0) = 33.00, \ldots, f(t_{10}) = f(5.0) = 49.00$. The total segment goes from $a = 0.0$ to $b = 5.0$, and we have $n = 10$ intervals. We write the width of an interval, or the **change in *t***, as $\boldsymbol{\Delta t} = \dfrac{b-a}{n} = \dfrac{5-0}{10} = 0.5$ sec. For this

example, the underestimate is a **left-hand sum**, where we use the velocity value on the left of each interval, as follows:

$$\begin{aligned} \text{left-hand sum} &= 24.00(0.5) + 28.75(0.5) + 33.00(0.5) + \cdots + 48.75(0.5) \\ &= f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_9)\Delta t \end{aligned}$$

Similarly, the overestimate is a **right-hand sum**, where we use the velocity value on the right of each interval, as follows:

$$\begin{aligned} \text{right-hand sum} &= 28.75(0.5) + 33.00(0.5) + \cdots + 48.75(0.5) + 49.00(0.5) \\ &= f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_9)\Delta t + f(t_{10})\Delta t \end{aligned}$$

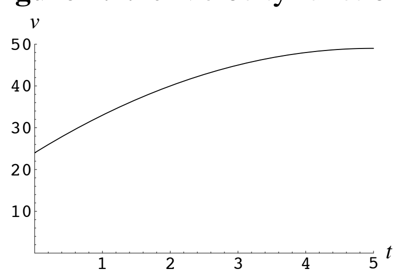
Perhaps we know the function f from a model or by estimation from the data. For this particular function, if the number of intervals approaches infinity with Δt going to 0, the left- and right-hand sums approach $203\frac{1}{3}$. Thus, the total distance traveled from $t = 0$ to $t = 5$ sec, is $203\frac{1}{3}$-m, and as the figures indicate, $203\frac{1}{3}$ is also the area under the velocity curve in Figure 2.4.10.

Table 2.4.4 Table 2.4.3 with notation

	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_1
t (sec)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.

	$f(t_0)$	$f(t_1)$	$f(t_2)$	$f(t_3)$	$f(t_4)$	$f(t_5)$	$f(t_6)$	$f(t_7)$	$f(t_8)$	$f(t_9)$	$f(t_{10})$
v (m/sec)	24.00	28.75	33.00	36.75	40.00	42.75	45.00	46.75	48.00	48.75	49.00

Figure 2.4.10 Velocity function with t in sec and v in m/sec



Quick Review Question 3 Using Quick Review Question 2, give values for the following:

a. a
b. b
c. n
d. Δt
e. Times $t_0, t_1, \ldots, t_n$
f. Velocities $f(t_0), f(t_1), \ldots, f(t_n)$, where $v = f(t)$ is the velocity function

Definite Integral

The limit of such a sum has so many applications other than computation of the total distance from velocity and the area under the curve that it has a special name and notation, the **definite integral**, as follows:

$$\int_a^b f(t)dt = \lim_{n\to\infty}(\text{left - hand sum}) = \lim_{n\to\infty}(f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t)$$

and

$$\int_a^b f(t)dt = \lim_{n\to\infty}(\text{right - hand sum}) = \lim_{n\to\infty}(f(t_1)\Delta t + f(t_1)\Delta t + \cdots + f(t_n)\Delta t),$$

where the width of an interval $\Delta t = (b - a)/n$ gets smaller as n gets larger.

Definitions If f is continuous (unbroken) for $a \le t \le b$, then the **left-hand sum** is

left-hand sum = $f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t$

and the **right-hand sum** is

right-hand sum = $f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t$

where $\Delta t = (b - a)/n$. The left- and right-hand sums are called **Reimann sums**.

Definition The **definite integral** of f from a to b is

$$\int_a^b f(t)dt = \lim_{n\to\infty}(\text{left - hand sum}) = \lim_{n\to\infty}(f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t)$$

and

$$\int_a^b f(t)dt = \lim_{n\to\infty}(\text{right - hand sum}) = \lim_{n\to\infty}(f(t_1)\Delta t + f(t_1)\Delta t + \cdots + f(t_n)\Delta t)$$

The function f is the **integrand**, and a and b are the **upper** and **lower limits of integration**, respectively.

Quick Review Question 4 Suppose $v = f(t)$ is the continuous (unbroken) velocity function for the cyclist in Quick Review Question 2. Give the following:

a. The definite integral for the total distance the cyclist travels during the indicated time up the mountain

b. The definite integral for the area under the curve during that period

c. The limits of integration

Total Change

Above, we saw that the definite integral of a velocity function from time $t = 0.0$ to time $t = 5.0$ sec gives the total change in distance during that period. Velocity is the rate of change of position with respect to time, so the definite integral of the rate of change of position yields the total change in position. In general, the definite integral of a rate of change of a function is the total change in that function. Because a rate of change is a derivative, the following expresses this fact symbolically:

$$\int_a^b F'(t)dt = \begin{pmatrix} \text{total change in } F(t) \\ \text{from } t = a \text{ to } t = b \end{pmatrix} = F(b) - F(a)$$

For example, suppose the instantaneous rate of change of the number of **disintegrations per minute** (**dpm**) per gram (A) of radioactive carbon-14 in a gram of a dead tree is $dA/dt = -15.3\, e^{-0.000121t}$ dpm/g/year from the time t the tree dies (Higham; Mahaffy). An estimate of the total change in the number of particles of carbon-14 between years 10 and 20 is as follows:

$$\int_{10}^{20} A'(t)dt = \int_{10}^{20}(-15.3e^{-0.000121t})dt = A(20) - A(10)$$

Using a computational tool that integrates or knowledge of integration, we can calculate the answer as approximately -152.723 dpm/g. From year 10 to year 20, a gram of carbon from the dead tree loses about 153 dpm per gram of carbon-14.

Quick Review Question 5 Suppose at time t = 5 hr the rate of change of a population of bacteria is $417(2^t)$.

a. Give the appropriate notation to estimate the increase in the number of bacteria from t = 5 hr to t = 9 hr.

b. Using the answer from Part a in a computational tool that integrates, we can calculate this increase in the population as about 288,770 bacteria. If the number of bacteria at time t = 5 hr is about 18,650, estimate the number of bacteria at time t = 9 hr.

Fundamental Theorem of Calculus

We have observed that the definite integral of a rate of change, or derivative, of a function gives the total change in that function. This result is the essential connection between differential and integral calculus, and the name of the theorem that explicitly states the relationship, **The Fundamental Theorem of Calculus**, indicates its significance.

The Fundamental Theorem of Calculus If f is continuous (unbroken) on the interval from a to b and $f(t) = F'(t)$ is the derivative, or rate of change, of F with respect to t, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

or

$$\int_a^b F'(t)dt = F(b) - F(a)$$

That is, the definite integral of a derivative, or a rate of change, of a function is the total change in the function from the lower limit of integration to the upper.

If the derivative of F is f, or $F'(t) = f(t)$, we call the function F an **antiderivative** of f. For example, we can show that one antiderivative of the velocity function $f(t) = -t^2 + 10t + 24$, whose graph is in Figure 2.4.10, is $F(t) = -t^3/3 + 5t^2 + 24t$. An infinite number of antiderivatives exist for $f(t) = -t^2 + 10t + 24$, because the derivative of $-t^3/3 + 5t^2 + 24t + C$, where C is any constant, is also $-t^2 + 10t + 24$. Thus, $-t^3/3 + 5t^2 + 24t + 1$, $-t^3/3 + 5t^2 + 24t + 37.8$, $-t^3/3 + 5t^2 + 24t - 3$, etc. are all antiderivatives of $-t^2 + 10t + 24$. We call the most general antiderivative of $f(t) = -t^2 + 10t + 24$, namely $-t^3/3 + 5t^2 + 24t + C$ for arbitrary constant C, the **indefinite integral** of f and employ a similar notation to that of the definite integral, as follows:

$$\int (-t^2 + 10t + 24)\ dt = -\frac{t^3}{3} + 5t^2 + 24t + C$$

Definition F is an **antiderivative** of f if $F'(t) = f(t)$, or the derivative of F is f.

Definition The **indefinite integral** of $f(t)$ is $F(t) + C$, where $F'(t) = f(t)$ and C is an arbitrary constant. The notation for the indefinite integral is as follows:

$$\int f(t)dt = F(t) + C$$

Quick Review Question 6 The derivative of $3x^6$ is $18x^5$. Using these functions, complete the following statements:

a. _________ is an antiderivative of _________.

b. $\int$ ____________ = ___________.

The velocity function $f(t) = -t^2 + 10t + 24$ is a rate of change, or derivative, of position with respect to time. The definite integral of f from $t = 0$ to $t = 5$ hr is the total change in the position. $F(t) = -t^3/3 + 5t^2 + 24t$ is an antiderivative of f, or $F'(t) = f(t)$. Thus,

$$\int_0^5 (-t^2 + 10t + 24)\, dt = F(5) - F(0)$$

We substitute 5 for t in $F(t)$ and subtract the substitution of 0 for t in $F(t)$, as follows:

$$F(5) - F(0) = (-(5^3)/3 + 5(5^2) + 24(5)) - (-(0^3)/3 + 5(0^2) + 24(0)) = 203\tfrac{1}{3}$$

This value is the total change in position indicated above.

To recap, if a function f has an antiderivative F, then to calculate the definite integral of f from a to b, we compute $F(b) - F(a)$. However, not all functions have antiderivatives. In such cases, we estimate the definite integral using a technique of **numeric integration**. We consider several such methods in the text. Many computational tools employ numeric integration techniques in their computations of definite integrals.

Quick Review Question 7 Using the fact that $3x^6$ is an antiderivative of $18x^5$, compute

$$\int_1^2 18x^5 dx$$

Differential Equations Revisited

Module 2.3 on "Rate of Change" defines a differential equation as an equation that contains a derivative. For example, suppose y is the position of a bicyclist at time t, and the following differential equation for the rate of change of y with respect to t gives the bicyclist's velocity function:

$$dy/dt = -t^2 + 10t + 24$$

We take the indefinite integral to find a general position function, as follows:

$$\int (-t^2 + 10t + 24)\, dt = -\frac{t^3}{3} + 5t^2 + 24t + C$$

Thus, the general position function is $y = -t^3/3 + 5t^2 + 24t + C$. We must have additional information to determine C. Frequently, we know the initial value of the function. For example, suppose initially, at time $t = 0$, the bicyclist is at position $y_0 = 30$ km from a starting location. Substituting 0 for t and 30 for y, we obtain a specific solution, namely, $y = -t^3/3 + 5t^2 + 24t + 30$, to the differential equation.

Quick Review Question 8 Using the fact that $\int 18x^5 dx = 3x^6 + C$ from Quick Review Question 7, solve the differential equation $dy/dx = 18x^5$ with initial condition $y_0 = 14$.

Exercises

$\boxed{\textbf{1.}}$ Suppose someone standing on a bridge throws a ball straight up over the water. With up being positive, suppose the velocity function for the ball is $v(t) = 15 - 9.8t$ in m/sec.

a. When is the velocity zero? At this instant, the ball is at its highest point.

b. Over what time period is the ball going up?

c. Generate a table of values from $t = 0$ to $t = 4$ similar to Table 2.4.3 with $\Delta t = 0.5$.

$\boxed{\textbf{d.}}$ Using these values, under- and overestimate the total change in position (height) from $t = 0$ to $t = 1.5$ sec.

e. Using these values, under- and overestimate the total change in position from $t = 2$ to $t = 4$ sec. Why is your result negative?

$\boxed{\textbf{f.}}$ Use a computational tool that integrates or an integration formula from calculus to solve the differential equation $dy/dt = 15 - 9.8t$ with initial condition $y_0 = 11$ m for the position (height) function $y(t)$. The solution is the height function whose graph is in Figure 2.3.4 of the module "Fundamental Concepts of Differential Calculus."

g. Using the position function in Part f, determine when the ball hits the water. At this instant, the position of the ball is at 0 m.

h. Graph the velocity function from time $t = 0$ to $t = 4$ sec.

$\boxed{\textbf{i.}}$ Using the formula for the area of a triangle and your answer from Part a, determine the area under the velocity curve from $t = 0$ sec to the time at which the velocity is zero. The area of a triangle is $0.5bh$, where b is the base and h is the height.

$\boxed{\textbf{j.}}$ Using your answer from Part i, determine the total change in position of the ball from the time it is thrown until the time it reaches its highest point.

k. Using the formula for the area of a triangle (see Part i) and your answers from Parts a and g, determine the area between the velocity curve and the t-axis from the time at which the ball is at its highest point until it hits the water.

l. Using your answer from Part k, determine the total change in position of the ball from the time at which the ball is at its highest point until it hits the water. Why should your answer be negative?

m. Using your answers from Parts j and l, determine the ball's total change in position from $t = 0$ until it hits the water. How is your answer related to the initial condition $y_0 = 11$ m?

n. Using your answer for the position function from Part f, determine the ball's total change in position from $t = 0$ until it hits the water. Do your answers from this and Part m agree?

2. Use the facts that 1 meter = 3.281 feet and 1 mile = 5280 feet, to compute the following.

a. The velocity 65 km/hr from Figure 2.4.1 in miles per hour (mph)

b. The velocities 70 km/hr and 60 km/hr from Quick Review Question 1 in mph
c. The velocities 24 m/sec and 49 m/sec from Table 2.4.1 in ft/sec
d. The total distance traveled ($203\frac{1}{3}$ m) for that example in feet
e. The velocities 80.5 km/hr and 25.1 km/hr from Quick Review Question 2 in mph
f. Your answer for an underestimate total distance traveled from Quick Review Question 2a in miles

3. For Quick Review Question 5, use a computational tool that integrates or an integration formula from calculus to obtain the increase in the number of bacteria from t = 5 hr to t = 9 hr.

4. Suppose that Q is the total quantity of salt in pounds in a reservoir. During a certain period of time, the amount of salt is increasing due to runoff from rains at the rate $dQ/dt = 10e^{-0.01t}$ pounds/day.
a. Generate a table of values from t = 100 to t = 250 days similar to Table 2.4.3 with Δt = 50 days.
b. Using these values, under- and overestimate the total change in salt from t = 100 to t = 250 days.
c. Repeat Part a using Δt = 25 days.
d. Repeat Part b using Δt = 25 days.
e. Use integration with an appropriate computational tool or an integration formula from calculus to determine the total change in salt from t = 100 to t = 250 days.
f. Repeat Part e for t = 0 to t = 250 days.
g. Use a computational tool that integrates or knowledge of calculus to solve the differential equation with initial condition Q_0 = 0 pounds.

Project

1. Using a computational tool that integrates, develop a document to explain and illustrate the material of this module. Use different functions than appear in the module for your examples.

Answers to Quick Review Questions

1. 70(0.5) + 60(1.5) = 125 km. The velocity is multiplied by the length of time for each segment.

2. **a.** (68.1)(0.5) + (44.9)(0.5) + (30.1)(0.5) + (25.1)(0.5) = 84.1 km. Each interval lasts 0.5 hours. There are four half-hour periods, so the sum consists of four terms. Because the function is decreasing, we use the velocities at the end (on the right) of each interval for the underestimate.
b. (80.5)(0.5) + (68.1)(0.5) + (44.9)(0.5) + (30.1)(0.5) = 111.8 km. Because the function is decreasing, we use the velocities on the left of each of the four intervals for the overestimate.

3. **a.** $a = 1$
b. $b = 3$
c. $n = 4$
d. $\Delta t = 0.5$

e. $t_0 = 1.0, t_1 = 1.5, t_2 = 2.0, t_3 = 2.5, t_4 = 3.0$
f. $f(t_0) = 80.5, f(t_1) = 68.1, f(t_2) = 44.9, f(t_3) = 30.1, f(t_4) = 25.1$

4. **a.** $\int_1^3 f(t)dt$
b. $\int_1^3 f(t)dt$
c. 1 and 3

5. **a.** $\int_5^9 417(2^t)dt$
b. 307,420 bacteria

6. **a.** $3x^6$ is an antiderivative of $18x^5$
b. $\int 18x^5dx = 3x^6 + C$

7. $189 = 3(2)^6 - 3(1)^6$

8. $y = 3x^6 + 14$

References

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